Principal Angles in Higher Dimensions

Beier Zhu

Definition 1. *Given two non-empty subspaces R and S of $\mathbb{R}^d$, where $r = \min(\dim(R), \dim(S))$, we have r principal angles:*

$$0 \leq \theta_1 \leq ... \leq \theta_r \leq \pi/2. \tag{1}$$

The directions of the inequalities swap when we take the cosine of the principal angles:

$$1 \geq \cos\theta_1 \geq ... \geq \cos\theta_r \geq 0. \tag{2}$$

The cosines of the principal angles are given by the SVD – let $E \in \mathbb{R}^{d\times \dim(R)}$ and $F \in \mathbb{R}^{d\times \dim(S)}$ have orthonormal columns which span R and S respectively. Then we have:

$$\cos\theta_i = \sigma_i(E^T F), \tag{3}$$

where σ_i denotes the i-th largest singular value. In this paper, we are interested in the cosine of the largest angle between them, given by:

$$\cos\theta_{\max}(R, S) = \cos\theta_r, \tag{4}$$

Proposition 1. *Suppose $\dim(R) \leq \dim(S)$, and let $F \in \mathbb{R}^{d\times \dim(S)}$ have orthonormal columns that forms a basis for S. We have:*

$$\cos\theta_{\max}(R, S) = \min_{\mathbf{x}\in R, \|\mathbf{x}\|_2=1} \|F^T(\mathbf{x})\|_2 \tag{5}$$

Proof. Let $E \in \mathbb{R}^{d\times \dim(R)}$ and $F \in \mathbb{R}^{d\times \dim(S)}$ have orthonormal columns that span R and S respectively. Since $\dim(R) \leq \dim(S)$ (a crucial condition!), $F^T E$ is a "tall" matrix (it has more rows than columns). Let $r = \dim(R)$ and $\mathbf{v}_1, ..., \mathbf{v}_r$ be an orthogonal basis for $F^T E$ with eigenvalues $\sigma_i, i \in [r]$, then we can expand $\mathbf{x} \in R$ in this basis as:

$$\mathbf{x} = c_1\mathbf{v}_1 + ... + c_r\mathbf{v}_r \tag{6}$$

Denote $A = F^T E$, then we have

$$\begin{aligned}
\|A\mathbf{x}\|_2^2 &= (A\mathbf{x})^T A(c_1\mathbf{v}_1 + ... + c_r\mathbf{v}_r) && (7)\\
&= (A\mathbf{x})^T (c_1\sigma_1\mathbf{v}_1 + ... + c_r\sigma_r\mathbf{v}_r) && (8)\\
&= c_1^2\sigma_1^2\mathbf{v}_1^2 + ... + c_r^2\sigma_r^2\mathbf{v}_r^2 && (9)\\
& && (10)
\end{aligned}$$

Since σ_r is the smallest singular value and $\|\mathbf{x}\|_2 = 1$, we get

$$\|A\mathbf{x}\|_2^2 \geq \sigma_r^2(c_1^2\mathbf{v}_1^2 + ... + c_r^2\mathbf{v}_r^2) = \sigma_r^2\|\mathbf{x}\|^2 = \sigma_r^2 \tag{11}$$

So we have

$$\sigma_{\min}(F^T E) = \min_{\|\mathbf{x}\|_2=1} \|F^T E\mathbf{x}\|_2 \tag{12}$$

The result now follows from some algebra:

$$\begin{aligned}
\cos\theta_{\max}(R, S) &= \sigma_{\min}(F^T E) && (13)\\
&= \min_{\|\mathbf{x}\|_2=1} \|F^T E\mathbf{x}\|_2 && (14)\\
&= \min_{\mathbf{x}\in R, \|\mathbf{x}\|_2=1} \|F^T(\mathbf{x})\|_2 && (15)
\end{aligned}$$

$\square$