Pinsker's Inequality

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Lemma 1 (**Log-Sum Inequality**). *For $a_i \geq 0$ and $b_i > 0$, then*

$$\sum_i a_i \log \frac{a_i}{b_i} \geq \left(\sum_i a_i\right) \log \frac{\sum_i a_i}{\sum_i b_i}. \tag{1}$$

Proof. Let $f(x) = x \log x$, which is convex since $f''(x) = \frac{1}{x} > 0$ for $x > 0$. Set $\lambda_i = \frac{b_i}{\sum_j b_j}$ and $x_i = \frac{a_i}{b_i}$. By Jensen's inequality,

$$\sum_i \lambda_i f(x_i) \geq f\left(\sum_i \lambda_i x_i\right). \tag{2}$$

Substituting gives Eq. (1). □

Theorem 1. *For discrete probability distributions $p = (p_i)$ and $q = (q_i)$,*

$$D_{\mathsf{TV}}(p, q) = \frac{1}{2} \sum_i |p_i - q_i| = \frac{1}{2} \|p - q\|_1, \quad D_{\mathsf{KL}}(p||q) = \sum_i p_i \log \frac{p_i}{q_i}. \tag{3}$$

Then,

$$D_{\mathsf{TV}}(p, q)^2 \leq \frac{1}{2} D_{\mathsf{KL}}(p||q) \tag{4}$$

The inequality also holds for continuous distributions. For simplicity and clarity, we present the proof for the discrete case.

Proof. Let

$$A = \{i : p_i \geq q_i\}, \quad a = \sum_{i \in A} p_i, \quad b = \sum_{i \in A} q_i. \tag{5}$$

Then $D_{\mathsf{TV}}(p, q) = a - b$. By Log-Sum Inequality, we have

$$\sum_{i \in A} p_i \log \frac{p_i}{q_i} \geq a \log \frac{a}{b}, \quad \sum_{i \notin A} p_i \log \frac{p_i}{q_i} \geq (1 - a) \log \frac{1 - a}{1 - b}. \tag{6}$$

Then,

$$D_{\mathsf{KL}}(p||q) \geq a \log \frac{a}{b} + (1 - a) \log \frac{1 - a}{1 - b}. \tag{7}$$

For fixed b, define

$$g(a) = a \log \frac{a}{b} + (1 - a) \log \frac{1 - a}{1 - b} - 2(a - b)^2.$$

We have $g(b) = 0, g'(b) = 0$. Moreover,

$$g''(a) = \frac{1}{a} + \frac{1}{1 - a} - 4 = \frac{1}{a(1 - a)} - 4 \geq 0,$$

since $a(1 - a) \leq \frac{1}{4}$. Thus, g is convex and attains its minimum at $a = b$. Therefore,

$$a \log \frac{a}{b} + (1 - a) \log \frac{1 - a}{1 - b} \geq 2(a - b)^2.$$

Substituting gives Eq. (4). □