Moment-Generating Function

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Definition 1. ***(Moment-Generating Function, MGF).*** *For a real-valued random variable X, the moment-generating function $M_X(\lambda)$ is defined as:*

$$M_X(\lambda) = \mathbb{E}[\exp(\lambda X)] \tag{1}$$

Lemma 1. ***(MFG of a Gaussian R.V.)*** *The moment-generating function of a Gaussian r.v. $X \sim \mathcal{N}(\mu, \sigma^2)$ is*

$$M_X(\lambda) = \exp(\lambda\mu + \frac{\sigma^2\lambda^2}{2}). \tag{2}$$

Proof. The p.d.f. for $\mathcal{N}(\mu, \sigma^2)$ is given by:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2}) \tag{3}$$

The MFG is computed as:

$$\mathbb{E}[\exp(\lambda X)] = \int_{-\infty}^{\infty} \exp(\lambda x) \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2}) \mathrm{d}x \tag{4}$$

The key point is this integral can be re-written as the integral of the p.d.f of another Gaussian r.v.:

$$\begin{aligned}
\mathbb{E}[\exp(\lambda X)] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{x^2 - 2x(\mu+\sigma^2\lambda) + \mu^2}{2\sigma^2}) \mathrm{d}x & (5)\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{x^2 - 2x(\mu+\sigma^2\lambda) + (\mu+\sigma^2\lambda)^2 - (\mu+\sigma^2\lambda)^2 + \mu^2}{2\sigma^2}) \mathrm{d}x & (6)\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-(\mu+\sigma^2\lambda))^2}{2\sigma^2} + \frac{\sigma^4\lambda^2 + 2\mu\sigma^2\lambda}{2\sigma^2}) \mathrm{d}x & (7)\\
&= \exp(\mu\lambda + \frac{\sigma^2\lambda^2}{2}) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-(\mu+\sigma^2\lambda))^2}{2\sigma^2}) \mathrm{d}x & (8)
\end{aligned}$$

Note that the integrand is the p.d.f of $\mathcal{N}(\mu + \sigma^2\lambda, \sigma)$ r.v., and hence the integral equals to 1. This leaves Eq. 4. □

1 Application of MGF

1.1 Computation of Moments

The moment-generating function (MGF) is named as such because it serves the fundamental purpose of "generating" the moments of a random variable. It allows for the convenient calculation and extraction of all the moments (i.e., expected values of powers) of a random variable.

Take a power series expansion of the MGF:

$$\begin{aligned}
M_X(\lambda) = \mathbb{E}[\exp(\lambda X)] = \mathbb{E}[\sum_{n=0}^{\infty} \frac{(\lambda X)^n}{n!}] &= \sum_{n=0}^{\infty} \mathbb{E}[\frac{(\lambda X)^n}{n!}] & (9)\\
&= 1 + \lambda\mathbb{E}[X] + \frac{\lambda^2}{2!}\mathbb{E}[X^2] + \frac{\lambda^3}{3!}\mathbb{E}[X^3] + \cdots & (10)
\end{aligned}$$

The n-th moments $\mathbb{E}[X^n]$ can be computed as:

$$\mathbb{E}[X^n] = \frac{\mathrm{d}M_X(\lambda)}{\mathrm{d}\lambda^n}\Big|_{\lambda=0} \tag{11}$$

Example 1. The 1st to 4th moments of a Gaussian distribution using the MGF are:

$$\mathbb{E}[X] = \mu \tag{12}$$
$$\mathbb{E}[X^2] = \mu^2 + \sigma^2 \tag{13}$$
$$\mathbb{E}[X^3] = \mu^3 + 3\mu\sigma^2 \tag{14}$$
$$\mathbb{E}[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 \tag{15}$$

Example 2. (Mean and variance of $\chi^2(k)$) A distribution of chi-squre χ^2 with k degrees of freedom is the distribution a sum of the squares of k independent standard normal r.v. Let $Z_1, Z_2, ..., Z_k$ be independent standard norm distribution, each distributed as $\mathcal{N}(0, 1)$. Then the r.v. X defined by:

$$X = \sum_{i=1}^{k} Z_i^2 \tag{16}$$

follows a chi-square distribution with k degrees of freedom, denoted as $X \sim \chi^2(k)$. The mean $\mathbb{E}[X] = k$ and the variance $\mathbb{V}[X] = 2k$.

With $\mathbb{E}[Z_i^2] = 1$ according to Eq. (13), we have

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{k} Z_i^2] = \sum_{i=1}^{k} \mathbb{E}[Z_i^2] = k \tag{17}$$

Since Z_i^2 are independent, the variance of X is the sum of the variance of Z_i^2:

$$\mathbb{V}[X] = \mathbb{V}[\sum_{i=1}^{k} Z_i^2] = \sum_{i=1}^{k} \mathbb{V}[Z_i^2] \tag{18}$$

With $\mathbb{E}[Z_i^4] = 3$ according to Eq. (15), the variance of Z_i^2 is:

$$\mathbb{V}[Z_i^2] = \mathbb{E}[Z_i^4] - (\mathbb{E}(X_i^2))^2 = 3 - 1^2 = 2. \tag{19}$$

Combining Eq. (19) with Eq. (18), we have:

$$\mathbb{V}[X] = 2k \tag{20}$$

1.2 Sum of Gaussians

Let X and Y be independent random variables that are normally distributed, then their sum is also normally distributed. i.e., if

$$X \sim \mathcal{N}(\mu_X, \sigma_X^2) \tag{21}$$
$$Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2) \tag{22}$$
$$Z = X + Y, \tag{23}$$

then

$$Z \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2) \tag{24}$$

Proof. The moment generating function of Z is given by:

$$\begin{aligned} M_Z(\lambda) &= \mathbb{E}[\exp(\lambda Z)] = \mathbb{E}[\exp(\lambda(X + Y))] & (25)\\ &= \mathbb{E}[\exp(\lambda X)]\mathbb{E}[\exp(\lambda Y)] = M_X(\lambda)M_Y(\lambda) & (26)\\ &= \exp(\lambda\mu_X + \frac{\sigma_X^2\lambda^2}{2})\exp(\lambda\mu_Y + \frac{\sigma_Y^2\lambda^2}{2}) & (27)\\ &= \exp(\lambda(\mu_X + \mu_Y) + \lambda^2(\sigma_X^2 + \sigma_Y^2)/2) & (28) \end{aligned}$$

This is the moment generating function of the normal distribution with the mean $\mu_X + \mu_Y$ and the variance $\sigma_X^2 + \sigma_Y^2$. □