Tail Sum and Intergal for Expectation

Beier Zhu

Proposition 1. ***(Tail Sum Formulation for Expectation, Discrete Random Variable)*** *For X with possible values $\{0, 1, ..., n\}$*

$$\mathbb{E}(X) = \sum_{j=1}^{n} \mathbb{P}(X \geq j) \tag{1}$$

Proof. Define $p_j = \mathbb{P}(X = j)$. Then the expectation $\mathbb{E}(X) = 1p_1 + 2p_2 + 3p_3 + ... + np_n$ is the following sum:

$$\begin{aligned} & p_1 + \\ & p_2 + p_2 + \\ & .... \\ & p_n + p_n + p_n + ... + p_n \end{aligned}$$

By the addition rule of probabilities, and the assumption that the only possible values of X are $\{0, 1, ..., n\}$, the sum of the first column of p's is $\mathbb{P}(X \geq 1)$, the sum of the second column is $\mathbb{P}(X \geq 2)$, and so on. The sum of j-th column is $\mathbb{P}(X \geq j)$, $1 \leq j \leq n$. The whole sum is the sum of the column sums. □

Proposition 2. ***(Tail Integral Formulation for Expectation, Continuous Random Variable)*** *Let X be a positive random variable with c.d.f F.*

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X \geq t)dt \tag{2}$$

Proof. Let function $x1[x > 0]$ has derivative $1[x > 0]$ everywhere except for $x = 0$, then

$$x1[x > 0] = \int_0^x 1[t > 0]dt = \int_0^\infty 1[x > t]dt, \forall x \in \mathbb{R}. \tag{3}$$

Applying this identity to a non-negative random variable X yields

$$X = \int_0^\infty 1[X > t]dt \tag{4}$$

Taking expectations of both sides and using Fubini to interchange integrals,

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X)XdX = \int_0^\infty \mathbb{P}(X) \int_0^\infty 1[X > t]dtdX \tag{5}$$

$$= \int_0^\infty \int_0^\infty \mathbb{P}(X)1[X > t]dXdt \tag{6}$$

$$= \int_0^\infty \mathbb{P}(X > t)dt \tag{7}$$

□